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LETTER TO THE EDITOR

Conduction in random networks of super-normal conductors: Geometrical interpretation and enhancement of nonlinearity

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Abstract. Critical behaviour of conduction in random networks of super-normal conductors near the percolation threshold is investigated from a geometrical point of view. A fractal dimensionality d_{CA} describing a contact area between typical size clusters is introduced and evaluated explicitly. Non-analytic power law behaviour (divergence) of the conductivity and dielectric constant is given a purely geometrical interpretation in terms of d_{CA}. Enhancement of nonlinearity in networks of super-nonlinear conductors is elucidated.

Percolation is a good model of disordered mixtures such as random resistor networks and dilute ferromagnets. When the difference between physical quantities of different species is large enough, the system exhibits a non-analytic power law behaviour near the percolation threshold p_c of good components, namely components with a larger physical quantity. In a network where two kinds of conductors with conductances g_A and g_B as $g_A/g_B \gg 1$ are randomly distributed with probabilities p and $1-p$, the conductivity σ of the whole network follows (Efros and Shklovskii 1976, Straley 1976, 1978)

$$\sigma \propto \varepsilon^t g_A \qquad (p > p_c,\ g_B/g_A \ll \varepsilon^{t+s}), \tag{1}$$

$$\sigma \propto |\varepsilon|^{-s} g_B \qquad (p < p_c,\ g_B/g_A \ll |\varepsilon|^{t+s}), \tag{2}$$

$$\sigma \propto g_A^{s/(t+s)} g_B^{t/(t+s)} \qquad (g_B/g_A \gg |\varepsilon|^{t+s}), \tag{3}$$

where $\varepsilon \equiv (p-p_c)/p_c \ll 1$ and t and s are non-integer critical exponents. Above p_c, good (A) conductors form an infinite percolation cluster. The fractal geometry of this infinite cluster yields the critical behaviour (1) and various unusual dynamics of percolating systems, which have extensively been studied by many authors (Alexander and Orbach 1982, Rammal and Toulouse 1983, Gefen *et al* 1983, Harris and Stinchcombe 1983, Ohtsuki and Keyes 1984a, b). Recently, we have shown that most of the anomalies are given a unified geometrical interpretation and described generally by the exponent t and other static exponents for percolation (Keyes and Ohtsuki 1984). The geometrical concepts obtained are useful not only to derive known results but also to predict new phenomena (Ohtsuki and Keyes 1984c).

Below p_c, on the other hand, typical size clusters (TSCs) of good conductors exist and dominate the behaviour (2). Thus, the exponent s is also considered to represent the fractal nature of TSCs and associated anomalous dynamics. Recently, Coniglio and Stanley (1984) have presented a geometrical picture of (2). In this letter, we discuss in some detail and clarify a geometrical meaning of not only (2) but also (3) and a power law divergence of a dielectric constant. Then enhancement of nonlinearity is predicted to occur from the geometrical point of view.

First, we consider conduction in superconductor–normal conductor networks ($g_A = \infty$, g_B is non-zero and finite) and discuss the critical behaviour (2). In ordinary d-dimensional systems, the conductance $G(L)$ of a hypercube of volume $v = L^d$ is in proportion to the number L^{d-1} of current-carrying channels and in inverse proportion to the length L of channels, i.e., $G(L) \propto L^{d-2}$ (Stauffer 1979). In the length scale larger than a coherence length ξ, the network is also homogeneous and

$$G(L) = (L/\xi)^{d-2} G(\xi). \tag{4}$$

Just below p_c, TSCs of superconductors of size ξ govern the conduction and $G(\xi)$ represents the conductance between adjacent TSCs. Neighbouring TSCs are connected by a slit of normal conductors. Since the resistance of TSCs is zero, $G(\xi)$ is expressed as

$$G(\xi) = \sum_l N(l) g(l), \tag{5}$$

where $N(l)$ is the number of channels of length l connecting adjacent TSCs and $g(l)$ is the conductance of the channel. The self-similarity of the system suggests that the distribution of l ranges from unity to ξ and $N(l)$ obeys a power law $N(l) \propto l^{-\eta}$ (Stauffer 1979). Obviously, $\eta > 0$ and $g(l)$ is in inverse proportion to l

$$g(l) = l^{-1} g_B. \tag{6}$$

The main contribution in the summation (5) comes from the lower bound $l = 1$ (Stauffer 1979). In other words, shortest paths composed of a single normal conductor dominate the conduction between TSCs (Straley 1980) and $G(\xi)$ is written as

$$G(\xi) \simeq N_c g_c, \tag{7}$$

where $N_c = N(1)$ and $g_c = g(1) = g_B$. Inserting (7) into (5) and comparing with (2) and $\sigma = G(L)/L^{d-2}$, we find

$$N_c \propto \xi^{d_{CA}}, \tag{8}$$

$$d_{CA} \equiv d - 2 + s/\nu, \tag{9}$$

where ν is the critical exponent for ξ defined by $\xi \propto |\varepsilon|^{-\nu}$ (Stauffer 1979).

The fractal dimensionality d_{CA} is equivalent to that d_u of the unscreened perimeter introduced by Coniglio and Stanley (1984). Here, however, we call d_{CA} a 'contact area' dimensionality, because d_{CA} rather describes the effective area of a contact region between TSCs. If TSCs are fully compact dense clusters, d_{CA} should coincide with $d-1$, the dimensionality of normal surfaces, but sparse ramified structure of TSCs causes d_{CA} to be less than $d-1$. On the other hand, TSCs have a fractal dimensionality $d_F = d - \beta/\nu$ (Kirkpatrick 1978). The interpenetration of TSCs also gives rise to a larger value of d_{CA} than that of $d_F - 1$. In addition, N_c is clearly less than ξ^{d_F}, the number of all sites (bonds) belonging to a TSC. As a result, we propose inequalities

$$d - 1 \geqslant d_{CA}, \qquad d_F \geqslant d_{CA} \geqslant d_F - 1, \tag{10}$$

or equivalently,

$$1 \geqslant s/\nu, \qquad 2 \geqslant (s+\beta)/\nu \geqslant 1. \tag{11}$$

Table 1 shows explicit values of d_{CA} evaluated from available data of ν, β and s. The

Table 1. Contact area dimensionality d_{CA}.

d	ν^a	β^a	s^b	d_F	d_{CA}
1	1	0	1	1	0
2	1.33[c]	0.14[d]	1.29[e]	1.9	0.97
3	0.88[f]	0.4 ~0.45[g]	0.7	2.5	1.8
4	0.7	0.5	0.6	3.3	2.8_5
5	0.6	0.7		3.8	
6	0.5	1	0	4	4

[a] Stauffer (1979).
[b] Straley (1978).
[c] den Nijs (1979).
[d] Pearson (1980), Nienhuis *et al* (1980).
[e] Hong *et al* (1984).
[f] Heermann and Stauffer (1981).
[g] Margolina *et al* (1982), Gaunt and Sykes (1983).

inequalities (10) and (11) are satisfied in all dimensions. At $d=6$, $d_{CA}=d_F$ and almost all sites in a TSC are linked to other TSCs by a single conductor. For $d=2$, in contrast, d_{CA} is nearly equal to $d-1$. This reflects rather dense structure of TSCs at low dimensions. Except at $d=1$, however, the equality $s/\nu=1$ is considered not to hold generally. Thus, a hyperscaling relation $t+s=2\nu$ suggested by Straley (1980) fails at $d=2$ where $t=s$ (Straley 1977). We conclude that the non-analytic behaviour (2) and the non-integer exponent s stem from the fractal nature of the contact region between TSCs.

Next, we investigate the critical behaviour (3). In a network where both g_A and g_B are non-zero and finite, there exist two kinds of conduction mechanisms; one is the conduction along clusters of good bonds and the other is that through bad conductors. The conductance $G(L)$ becomes a sum of conductances G_A and G_B due to the former and the latter mechanism, respectively. In the length scale less than ξ, $L \lesssim \xi$, equations (1) and (2) together with scaling arguments lead to $G_A(L) \propto L^{d-2-t/\nu} g_A$ and $G_B(L) \propto L^{d-2+s/\nu} g_B$. When $g_B/g_A \ll |\varepsilon|^{t+s}$, $G_A(L) \gg G_B(L)$ in this region. Hence, $G \simeq G_A$ and σ varies as (1) above p_c, whereas below p_c, the resistance G_A^{-1} of TSCs is negligible and σ obeys (2) as shown before. In the case $1 \gg g_B/g_A \gg |\varepsilon|^{t+s}$, a characteristic length λ where $G_A(\lambda) \simeq G_B(\lambda)$ is defined by

$$\lambda \propto (g_A/g_B)^{\nu/(t+s)}. \tag{12}$$

When $L \lesssim \lambda$, G_A has the main contribution and

$$G(\lambda) \simeq G_A(\lambda) \propto \lambda^{d-2-t/\nu} g_A. \tag{13}$$

On the contrary, the contribution from G_B is dominant at $L \gtrsim \lambda$. Since the number of bad conductors is in proportion to the volume $v=L^d$, bad conductors are regarded as forming a normal d-dimensional lattice. In this case, therefore, we have

$$G(L) = (L/\lambda)^{d-2} G(\lambda). \tag{14}$$

In other words, clusters of good conductors of size λ dominate the conduction in the whole network, instead of TSCs of size ξ. Substitution of (12) and (13) into (14) gives

(3). It becomes evident that the non-analytic power law (3) comes from the crossover of the dominant conduction mechanism.

Besides (2), the exponent s describes the divergence of an electric susceptibility (dielectric constant) χ in a resistor–capacitor network (Efros and Shklovskii 1976)

$$\chi \propto |\varepsilon|^{-s} C_0, \tag{15}$$

where C_0 is the capacitance of the capacitors. This divergence is also given a geometrical interpretation in terms of d_{CA}. In the homogeneous region, $L \gg \xi$, the energy $U(L)$ accumulated in a hypercube of volume $v = L^d$ is in proportion to the volume and

$$U(L) = (L/\xi)^d U(\xi), \tag{16}$$

where $U(\xi)$ is the energy accumulated in a TSC. The voltage $V(\xi)$ applied between adjacent TSCs is given by

$$V(\xi) = (\xi/L)V(L) = \xi E, \tag{17}$$

where $E = V(L)/L$ is an electric field. The capacitance of a channel of length l is in inverse proportion to l. Similarly to the case of the conductance, therefore, shortest paths composed of a single capacitor between TSCs are considered to dominate the accumulation of $U(\xi)$ and we have

$$U(\xi) = N_c C_c V_c^2, \tag{18}$$

$$C_c = C_0, \tag{19}$$

$$V_c = V(\xi) = \xi E, \tag{20}$$

where C_c is the capacitance of the shortest channel and V_c is the voltage applied to the channel. Substituting (8), (9) and (18)–(20) into (16), we can derive (15), because $\chi = U(L)/(E^2 L^d)$.

The singularly large voltage difference as (20) in the channel causes enhancement of nonlinearity in the conduction behaviour of the system. Consider a network composed of superconductors ($g = \infty$) and nonlinear conductors whose current (I)–voltage (V) characteristics are given by

$$I = \sum_n g_n V^n, \tag{21}$$

where g_n is the nth order conductance. When $L \gg \xi$, the current $I(L)$ flowing through the volume $v = L^d$ is in proportion to the number of channels and expressed as

$$I(L) = (L/\xi)^{d-1} I(\xi). \tag{22}$$

Again, we can expect that conduction along the shortest paths of a single nonlinear conductor between TSCs is dominant, because the nth order conductance of channels is in inverse proportion to their length to the nth power. Then $I(\xi)$ becomes

$$I(\xi) = N_c I_c \tag{23}$$

and I_c follows (21)

$$I_c = \sum_n g_n (V_c)^n. \tag{24}$$

In this case, the relations (17) and (20) hold, too. From (17), (20) and (22)–(24), we find

$$I(L)/L^{d-1} = \sum_n \sigma_n (V(L)/L)^n \qquad (25)$$

with the nth order conductivity σ_n

$$\sigma_n = \xi^{n-1+s/\nu} g_n. \qquad (26)$$

With increasing n, σ_n diverges more rapidly, that is, the enhancement of nonlinearity occurs near a percolation threshold. The analogy between electric and mechanical networks informs that elastic networks of rigid bonds—unharmonic bonds with an isotropic force constant show the same behaviour (de Gennes 1976, Feng and Sen 1984). Furthermore, the enhancement of nonlinearity is considered to be observed widely in random mixtures of super-nonlinear constituents, e.g., central-force elastic networks (Feng and Sen 1984), because a singularly large excitation is generally exerted on nonlinear components in such systems. In linear systems, the average physical quantity of the whole system is a monotonically increasing function of a fraction of good components. In nonlinear systems, in contrast, we may expect complex behaviour due to enhancement of nonlinearity just below the threshold.

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